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TRANSVERSE STABILITY OF A LIQUID JET IN A COUNTERFLOWING AIR STREAM

V. M. Entov and A. L. Yarin UDC 532.521:532.522

The development of small perturbations which bend a liquid jet in a counterflowing air stream is analyzed here. It is demonstrated that the most dangerous perturbations of the jet axis have a spatial distribution, and the increment of perturbation buildup is calculated.

Straight jets of a liquid moving in air at sufficiently high velocities are unstable against transverse perturbations [1-4]. As a consequence, they acquire a waviness and eventually break up. The problem of dynamic action of an air stream on the a priori unknown surface of a jet with a flow also yet to be determined is a very difficult one and, for this reason, only the first steps have so far been taken toward a theoretical description of it [2, 5, 6].

Here will be analyzed the stability of a straight laminar jet of a viscous liquid against small long-wave spatial perturbations. The analysis will be based on the equations of dynamics of thin liquid jets [7], which in the case of small perturbations are

$$\begin{gathered}
\frac{\partial f}{\partial t}+f_0\frac{\partial V_\tau}{\partial s}=0,\\
\rho f_0\frac{\partial \mathbf{V}}{\partial t}=\frac{\partial}{\partial s}(P\boldsymbol{\tau}+\mathbf{Q})+\mathbf{q},\\
\rho\frac{\partial \mathbf{K}}{\partial t}=\frac{\partial \mathbf{M}}{\partial s}+\boldsymbol{\tau}\times\mathbf{Q},\\
\mathbf{K}=I(\mathbf{n}\Omega_n+\mathbf{b}\Omega_b),\quad \Omega_n=-V_{b,s}-\varkappa V_n,\quad \Omega_b=V_{n,s}-\varkappa V_b,\\
\mathbf{M}=3\mu I[\mathbf{n}(\Omega_{n,s}-\varkappa\Omega_b)+\mathbf{b}(\Omega_{b,s}+\varkappa\Omega_n)]-\alpha I a_0^{-1}k\mathbf{b}+\mathbf{M}_1,\\
P=3\mu f_0\frac{\partial V_\tau}{\partial s}+\pi a\alpha+\alpha f_0\frac{\partial^2 a}{\partial s^2}+P_1.
\end{gathered}\tag{1}$$

In the selected reference system the unperturbed jet remains at standstill while the air stream moves along its axis. The cross section of the jet is assumed to be circular, without body forces and rotation of the liquid about the jet axis ($\Omega_\tau = 0$).

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Perturbations of hydrostatic pressure in the surrounding air are transmitted to the liquid in the jet. Taking this into account gives rise to the M_1 terms and the P_1 term in the respective expressions for the moment of stresses and the longitudinal force in a cross section. We will calculate here the moment of stresses M_1 as well as the forces acting on the jet from the surrounding air, all needed for determining the transverse stability of such a jet. It ought to be noted that, owing to the inequality $\rho_1 \ll \rho$, only terms of the order $\rho_1 U_0^2$ contribute substantially to the determination of the dynamic action of air on the jet. The linear density of the moment of external forces **m** does not appear in the equation for the angular momentum, because in this case, it will be demonstrated here, m = 0.

Projecting the equations for the momentum and angular momentum (1) on the normal, the binormal, and the tangent to the jet axis, also considering the smallness of the components of velocity V and angular velocity Ω of the liquid, we obtain

$$
\begin{gathered}
2\frac{\partial a}{\partial t}+a_0\frac{\partial V_\tau}{\partial s}=0,\\
\rho f_0\frac{\partial V_\tau}{\partial t}=3\mu f_0\frac{\partial^2 V_\tau}{\partial s^2}+\pi\alpha\frac{\partial a}{\partial s}+\alpha f_0\frac{\partial^3 a}{\partial s^3}+\frac{\partial P_1}{\partial s}+q_\tau,\\
\rho f_0\frac{\partial V_n}{\partial t}=\frac{\partial Q_n}{\partial s}-\varkappa Q_b+\pi a_0 k\alpha+q_n,\\
\rho f_0\frac{\partial V_b}{\partial t}=\frac{\partial Q_b}{\partial s}+\varkappa Q_n+q_b,\\
-\rho I\frac{\partial^2 V_b}{\partial s\partial t}-\rho I\frac{\partial}{\partial t}(\varkappa V_n)=\frac{\partial M_n}{\partial s}-\varkappa M_b-Q_b,\\
\rho I\frac{\partial^2 V_n}{\partial s\partial t}-\rho I\frac{\partial}{\partial t}(\varkappa V_b)=\frac{\partial M_b}{\partial s}+\varkappa M_n+Q_n.
\end{gathered}
\tag{2}
$$

Here are retained only terms of first-order smallness, with the projections of those equations on the tangent τ being identically equal to zero.

The first two equations in system (2) describe the buildup of small axisymmetric perturbations in the jet, taking into account the effect of the air stream, and can be solved independently of the remaining equations (the increment γ_1 of buildup of axisymmetric perturbations has been determined by Weber [5]). Transverse perturbations of the jet axis are here insignificant. On the other hand, the remaining equations in system (2) describe small transverse perturbations of a liquid jet with variations of the radius disregarded. These perturbations are characterized by an increment γ, for which a corresponding dispersion equation will be derived here subsequently. When the maximum increment γ is much larger than the maximum increment γ_1, then transverse perturbations build up much faster than axisymmetric perturbations. In this case the radius of the jet can be assumed to remain constant.

We now proceed to determine the increment of buildup of small transverse perturbations in a jet of constant radius. First of all we calculate the aerodynamic force **q** and the moment M_1. For this we use the theory of motion of slender bodies ("fish") [8-10].

We introduce a Cartesian system of coordinates $O_1\xi\eta\zeta$, where the ξ axis coincides with the axis of the unperturbed jet and moves together with it at the velocity U_0 in the $\xi = -\infty$ direction. Let us parametrize the jet axis just as in study [7], namely ξ = s. The equations of perturbations of the jet axis will be

$$\eta = \mathrm{H}(s,\ t),\quad \zeta = \mathrm{Z}(s,\ t), \tag{3}$$

where H and Z, the displacements of the axis in directions $O_1\eta$ and $O_1\zeta$, respectively, are of the first-order smallness. Displacements of higher-order smallness will be disregarded. The equation of the jet surface is, moreover,

$$(\eta-\mathrm{H})^2+(\zeta-\mathrm{Z})^2=a_0^2\,. \tag{4}$$

The gas surrounding the jet will be regarded as an ideal and incompressible one, and its motion to be potential flow. The linearity of the problem allows us to represent the potential as a sum

$$\Phi = U_0 s + \varphi, \tag{5}$$

where the perturbation potential φ satisfies the equation

$$\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2} = 0, \tag{6}$$

with the necessary degree of accuracy, inasmuch as $\varphi_{,ss} \ll \varphi_{,\eta\eta}$, and $\varphi_{,ss} \ll \varphi_{,\zeta\zeta}$ in the case of long-wave perturbations of a slender jet. In other words, as is usually done in the theory of streamlining of thin bodies, the perturbed motion can be regarded as two-dimensional in every section normal to the velocity of the counterflowing stream [8-10]. At the jet surface there must be satisfied the condition of impermeability, which in the linearized form becomes

$$\begin{gathered} (\eta - \mathrm{H})\, v + (\zeta - Z)\, w = (\eta - \mathrm{H})\, V_\eta^* + (\zeta - Z)\, V_\zeta^*, \\ v = \frac{\partial \varphi}{\partial \eta}, \quad w = \frac{\partial \varphi}{\partial \zeta}, \quad V_\eta^* = D\mathrm{H}, \quad V_\zeta^* = DZ, \\ D = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial s}. \end{gathered} \tag{7}$$

Furthermore, perturbations must vanish at infinity

$$\varphi \to 0, \ \eta^2 + \zeta^2 \to \infty. \tag{8}$$

Changing in Eqs. (6)-(8) to polar coordinates r, θ ($\eta - \mathrm{H} = r \sin \theta, \zeta$ and $Z = r \cos \theta$), we have the potential problem

$$\begin{gathered} \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} = 0, \\ \frac{\partial \varphi}{\partial r} = V_\eta^* \sin \theta + V_\zeta^* \cos \theta, \quad r = a_0, \\ \varphi \to 0, \quad r \to \infty. \end{gathered} \tag{9}$$

The solution to problem (9) is

$$\varphi = -\frac{a_0^2}{r} \left(V_\eta^* \sin \theta + V_\zeta^* \cos \theta \right) = -\frac{a_0^2 \left[V_\eta^* (\eta - \mathrm{H}) + V_\zeta^* (\zeta - Z) \right]}{(\eta - \mathrm{H})^2 + (\zeta - Z)^2}. \tag{10}$$

Inserting the potential defined by expressions (5) and (10) into the Lagrange–Cauchy integral yields, after terms of higher-order smallness have been discarded, the pressure distribution over the jet surface

$$p = p_\infty + \rho_1 \left[(\eta - \mathrm{H})\, DV_\eta^* + (\zeta - Z)\, DV_\zeta^* \right]. \tag{11}$$

With the aid of relation (11) we find the linear density of external forces on the jet

$$\mathbf{q} = -\rho_1 U_0^2 f_0 (\mathbf{j}\mathrm{H}_{,ss} + \mathbf{k} Z_{,ss}) \tag{12}$$

and also the moment

$$\mathbf{M}_1 = \rho_1 U_0^2 k l \mathbf{b}. \tag{13}$$

The linear density of the moment of external forces **m** is in this case obviously zero, since the direction of these forces is along the radius-vector in a cross section of the jet.

Therefore, all quantities in the equation of dynamics (2) of a liquid jet are given. With the aid of Eqs. (1) and expression (13), moreover, we find the components of the vector M (moment of stresses) in a jet section

$$M_n = 3\mu I\left[-\frac{\partial^2 V_b}{\partial s^2} - \frac{\partial}{\partial s}(\varkappa V_n) - \varkappa\frac{\partial V_n}{\partial s} + \varkappa^2 V_b\right], \tag{14}$$

$$M_b = 3\mu I\left[\frac{\partial^2 V_n}{\partial s^2} - \frac{\partial}{\partial s}(\varkappa V_b) - \varkappa\frac{\partial V_b}{\partial s} - \varkappa^2 V_n\right] - \frac{\alpha}{a_0} Ik\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right).$$

The linearity of the problem allows us to consider only spiral perturbation of the jet axis

$$\mathrm{H} = A\exp(\gamma t)\cos(\chi s/a_0), \quad \mathrm{Z} = B\exp(\gamma t)\sin(\chi s/a_0). \tag{15}$$

Its curvature and twist are, accordingly,

$$k = (\chi/a_0)^2\exp(\gamma t)\,[A^2\cos^2(\chi s/a_0) + B^2\sin^2(\chi s/a_0)]^{1/2},$$
$$\varkappa = \chi ABa_0^{-1}\,[A^2\cos^2(\chi s/a_0) + B^2\sin^2(\chi s/a_0)]^{-1}. \tag{16}$$

Using the linearized kinematic relation between velocity U of a point s on the jet axis and velocity V of a liquid particle at that point [7], viz.,

$$\mathbf{U} = \mathbf{V} - \tau(\mathbf{V}\cdot\mathbf{i}), \quad \mathbf{U} = \mathbf{j}\mathrm{H}_{,t} + \mathbf{k}\mathrm{Z}_{,t}, \tag{17}$$

we obtain with the aid of expressions (15)

$$V_n = -\gamma\exp(\gamma t)\,[A^2\cos^2(\chi s/a_0) + B^2\sin^2(\chi s/a_0)]^{1/2}, \quad V_b = 0. \tag{18}$$

Projecting relation (12) on a normal and a binormal to the jet axis yields, with the aid of expressions (15) and with the necessary accuracy, the relation

$$q_n = -\rho_1 U_0^2 f_0 \chi^2 a_0^{-2}\exp(\gamma t)\,[A^2\cos^2(\chi s/a_0) + B^2\sin^2(\chi s/a_0)]^{1/2}, \quad q_b = 0. \tag{19}$$

Inserting expressions (16), (18), and (19) into the last four of Eqs. (2) and into expressions (14) yields the equations for small spatial perturbations of a slender liquid jet

$$\begin{gathered}
-\rho f_0\frac{\partial V_n}{\partial t} + \frac{\partial Q_n}{\partial s} - \varkappa Q_b + \pi a_0\alpha k + q_n = 0,\\
\frac{\partial Q_b}{\partial s} + \varkappa Q_n = 0,\\
-\rho I\frac{\partial^2 V_n}{\partial s\partial t} + \frac{\partial M_b}{\partial s} + Q_n = 0,\\
\rho I\varkappa\frac{\partial V_n}{\partial t} - \varkappa M_b - Q_b = 0,\\
\frac{\partial^2 V_n}{\partial s^2} = \frac{M_b}{3\mu I} + \varkappa^2 V_n + \frac{\alpha k}{3\mu a_0}\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right).
\end{gathered} \tag{20}$$

Moreover,

$$M_n = 0. \tag{21}$$

The equations of jet deflection in a plane are obtained from Eqs. (20) for $\varkappa = Q_b = 0$, with the second and the fourth of these equations becoming identities. After M_b has been determined from the last of Eqs. (20), the fourth of Eqs. (20) yields the projection of the shearing force on a binormal

$$Q_b = \rho\varkappa I \frac{\partial V_n}{\partial t} - 3\mu\varkappa I \left(\frac{\partial^2 V_n}{\partial s^2} - \varkappa^2 V_n\right) + \frac{\alpha\varkappa k I}{a_0}\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right). \quad (22)$$

The third of these equations yields

$$Q_n = \rho I \frac{\partial^2 V_n}{\partial s \partial t} - 3\mu I \left[\frac{\partial^3 V_n}{\partial s^3} - \frac{\partial}{\partial s}(\varkappa^2 V_n)\right] + \frac{\alpha I}{a_0}\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right)\frac{\partial k}{\partial s}. \quad (23)$$

Inserting expressions (22) and (23) into the second of Eqs. (20), with relations (16) and (18) taken into account, results in an identity. The first of Eqs. (20), together with expressions (22) and (23), yields

$$-\rho f_0 \frac{\partial V_n}{\partial t} + \rho I \left(\frac{\partial^3 V_n}{\partial s^2 \partial t} - \varkappa^2 \frac{\partial V_n}{\partial t}\right) + 3\mu I \left[\varkappa^2 \left(\frac{\partial^2 V_n}{\partial s^2} - \varkappa^2 V_n\right) - \right.$$
$$\left. - \frac{\partial^4 V_n}{\partial s^4} + \frac{\partial^2}{\partial s^2}(\varkappa^2 V_n)\right] + \frac{\alpha I}{a_0}\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right)\left(\frac{\partial^2 k}{\partial s^2} - k\varkappa^2\right) + \pi\alpha a_0 k + q_n = 0. \quad (24)$$

Relations (16), (18), and (19) reduce the resulting expression to

$$[A^2 \cos^2(\chi s/a_0) + B^2 \sin^2(\chi s/a_0)]^2 \left[\gamma^2 \left(\frac{\rho f_0 a_0}{\chi} + \frac{\rho I \chi}{a_0}\right) + \frac{3\mu\chi^3 I}{a_0^3}\gamma + \pi\alpha\chi - \rho_1 U_0^2 \pi a_0 \chi - \frac{\alpha I \chi^3}{a_0^4}\left(1 - \frac{\rho_1 a_0 U_0^2}{\alpha}\right)\right] = 0. \quad (25)$$

From here we obtain the dispersion equation for small perturbations of a liquid jet

$$\gamma^2 + \frac{3}{4}\frac{\mu\chi^4}{\rho a_0^2}\gamma + \left(\frac{\alpha}{\rho a_0^3} - \frac{\rho_1 U_0^2}{\rho a_0^2}\right)\chi^2 = 0. \quad (26)$$

Here have been retained only the principal terms when $\chi \to 0$, inasmuch as the long-wave approximation is considered.

The dispersion equation (26) for velocities

$$U_0 > U_0^* = (\alpha/\rho_1 a_0)^{1/2} \quad (27)$$

has a root $\gamma > 0$ and, therefore, transverse perturbations of the jet are unstable. The condition of instability (27) follows from the results of another study [6] pertaining to a jet of a nonviscous fluid. When inequality (27) holds true, then Eq. (26) yields the dimensionless wave number χ_* of the perturbation which builds up fastest and also its largest increment γ_*

$$\chi_* = \left[\frac{8}{9}\frac{\rho a_0^2}{\mu^2}\left(\rho_1 U_0^2 - \frac{\alpha}{a_0}\right)\right]^{1/6}, \quad \gamma_* = \frac{(a_0\rho_1 U_0^2 - \alpha)^{2/3}}{(3\mu\rho a_0^4)^{1/3}}. \quad (28)$$

When the action of surface tension is negligible as compared to the dynamic action of air causing breakup of the jet, then the maximum increment of buildup γ_{1*} of axisymmetric perturbations [5] is much smaller than γ_* under conditions of the inequality

$$\mu^2 (\rho a_0^2 \rho_1 U_0^2)^{-1} \gg 1. \quad (29)$$

Accordingly, variations of the radius on a jet cross section can be disregarded in an analysis of small transverse perturbations in a jet of sufficiently viscous fluids. It is exactly in such a jet where increment (28) characterizes the transverse perturbation which builds up fastest. We will note that the phenomenon of instability of transverse perturbations was discovered by Henlein [1] during experiments with jets of castor oil under conditions satisfying inequality (29).

Since breakup of a jet discharging from a nozzle at a velocity U_0 is caused by the perturbation which builds up fastest and it occurs when the maximum deflection of the jet axis reaches the order of magnitude of λ_*, we find the length of the jet prior to breakup to be

$$L=\Delta\left[\frac{3\mu\rho a_0^2 U_0^3}{(\rho_1 U_0^2-\alpha/a_0)^2}\right]^{1/3}. \tag{30}$$

with $\Delta = \ln(\lambda_*/\delta_0)$.

The dispersion equation (26) determines the increment of plane as well as of spatial perturbations of the jet axis. Consequently, both build up at the same rate and breakup of a jet should have a spatial pattern. Photographs of a jet in two projections [2] confirm this conclusion.

In conclusion, we will make reference to indications which suggest that disregarding the viscosity of air around a jet leads to an overestimate of pressure perturbations at the jet surface [2, 11]. It can be taken into account, as has been done in the latter study [11], by replacing expression (11) with the expression

$$p=p_\infty+C\rho_1[(\eta-\mathrm{H})DV_\eta^*+(\zeta-\mathrm{Z})DV_\zeta^*], \tag{31}$$

where C ($0 < C < 1$) is an empirical constant. In all relations which follow from expression (11), accordingly, ρ_1 must be replaced with the product $C\rho_1$ and one of the consequences will be, for instance, a higher threshold velocity U_0^*.

NOTATION

V, velocity of the liquid on the jet axis, with projections V_n, V_b, and V_τ on, respectively, the normal **n**, the binormal **b**, and the tangent τ to the jet axis; Ω, angular velocity of a liquid particle in a jet cross section; M, moment of stresses in a jet cross section about its center; Q, shearing force in a cross section; $P\tau$, longitudinal force in a cross section; q, a linearly distributed force (aerodynamic force in this case) acting on the jet; **m**, linear density of the moment of external forces; U, velocity of point s on the jet axis; s, a parameter of the jet axis; t, time; f, area of a jet cross section ($f = \pi a^2$), f_0 and a_0, unperturbed state; I, moment of inertia of an unperturbed jet cross section ($I = \pi/4 \cdot a_0^4$); k, curvature of the jet axis; $\varkappa$, twist of the jet axis; μ, dynamic viscosity of the liquid; α, coefficient of surface tension in the liquid; ρ, liquid density; ρ_1, air density; U_0, velocity of the air stream (or the discharge velocity of the jet); ξ, η, ζ, axes of Cartesian coordinates with the respective unit vectors **i**, **j**, **k**; p, air pressure; p_∞ pressure at infinity; A and B, constant coefficients; χ, dimensionless wave number of transverse perturbations; γ and γ_1, increments of buildup of, respectively, transverse and axisymmetric perturbations (asterisks denote thier maximum values); λ_*, wavelengthof the perturbation which builds up fastest; δ_0, maximum deviation of the jet axis from a straight line at time $t = 0$; and L, length of the jet prior to breakup.

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